

CONTRACTING THE WEIERSTRASS LOCUS TO A POINT

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ABSTRACT. We construct an open substack $U \subset \mathcal{M}_{g,1}$ with the complement of codimension ≥ 2 and a morphism from U to a weighted projective stack, which sends the Weierstrass locus $\mathcal{W} \cap U$ to a point, and maps $\mathcal{M}_{g,1} \setminus \mathcal{W}$ isomorphically to its image. The construction uses alternative birational models of $\mathcal{M}_{g,1}$ and $\mathcal{M}_{g,2}$ from [8].

INTRODUCTION

Let $\mathcal{W} \subset \mathcal{M}_{g,1}$ denote the locus in the moduli stack of smooth one-pointed curves of genus g , consisting of (C, p) such that p is a Weierstrass point on C , i.e., $h^1(gp) \neq 0$. It is well known that \mathcal{W} is an irreducible divisor. In this paper we construct a rational map from $\mathcal{M}_{g,1}$ to a proper DM-stack with projective coarse moduli space, which contracts \mathcal{W} to a single point and maps $\mathcal{M}_{g,1} \setminus \mathcal{W}$ isomorphically to its image (see Theorem A below). This is partly motivated by the question whether the class of the closure of \mathcal{W} in $\overline{\mathcal{M}}_{g,1}$ generates an extremal ray (we do not solve this; however, see Prop. 2.4.6, Rem. 2.4.7 and the discussion below). Note that for small g some pointed Brill-Noether divisors were shown to generate extremal rays in the effective cone of $\overline{\mathcal{M}}_{g,1}$ in [9], [5] and [6].

The construction involves certain moduli stacks studied in [8]. Namely, in [8] we introduced and studied the moduli stack of curves with marked points (C, p_1, \dots, p_n) , where C is a reduced projective curve of arithmetic genus g , such that $h^1(a_1p_1 + \dots + a_np_n) = 0$ for fixed integer weights $a_i \geq 0$ such that $a_1 + \dots + a_n = g$ (we assume that the marked points are smooth and distinct). We denote this stack by $\mathcal{U}_{g,n}^{ns}(a_1, \dots, a_n)$. We showed that $\mathcal{U}_{g,n}^{ns}(a_1, \dots, a_n)$ can be realized as a quotient of an affine scheme by a torus action and studied the related GIT picture which leads to interesting projective birational models of $\mathcal{M}_{g,n}$. In particular, for $n = 1$ and $a_1 = g$ there is a unique nonempty GIT quotient stack $\overline{\mathcal{U}}_{g,1}^{ns}(g)$, obtained from $\mathcal{U}_{g,1}^{ns}(g)$ by deleting one point corresponding to the most singular cuspidal curve. Furthermore, $\overline{\mathcal{U}}_{g,1}^{ns}(g)$ is a closed substack in a weighted projective space (see Sec. 1.1 for details).

We start by considering the natural rational map

$$\text{for}_2 : \mathcal{U}_{g,2}^{ns}(g-1, 1) \dashrightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g) \quad (0.0.1)$$

given by forgetting the second marked point (more precisely, the map for_2 is regular on a certain open substack which is dense in the component corresponding to smoothable curves). Our main technical result is that (0.0.1) is regular on the open substack of (C, p_1, p_2) such that $h^1((g+1)p_1) = 0$, and that the divisor, defined by the condition $h^1(gp_1) \neq 0$, gets contracted to a point (see Prop. 1.2.2). Furthermore, we show that this point has trivial group of automorphisms. We derive from this the following result.

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Theorem A. *Assume that $g \geq 2$. The natural open embedding of stacks*

$$\mathcal{M}_{g,1} \setminus \mathcal{W} \hookrightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g)$$

extends to a regular morphism

$$\phi = \phi_g : U \rightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g),$$

for some open substack $U \subset \mathcal{M}_{g,1}$ containing $\mathcal{M}_{g,1} \setminus \mathcal{W}$ and such that $\mathcal{M}_{g,1} \setminus U$ has codimension ≥ 2 in $\mathcal{M}_{g,1}$. Furthermore, ϕ contracts $U \cap \mathcal{W}$ to a single point, which has no nontrivial automorphisms.

More precisely, the open substack U in the above Theorem consists of (C, p) such that $h^1((g+1)p) = 0$ and $h^0((g-1)p) = 1$.

We study the case $g = 2$ in more detail. In this case we get a more precise result involving a certain modular compactification of $\mathcal{M}_{2,1}$.

Recall that Smyth introduced in [10] the notion of an extremal assignment, which is a rule associating to each stable curve of given arithmetic genus some of its irreducible components (this rule should be stable under degenerations). For each extremal assignment \mathcal{Z} , Smyth considered the moduli stack $\overline{\mathcal{M}}_{g,n}(\mathcal{Z})$ of \mathcal{Z} -stable curves, i.e., pointed curves C for which there exists a stable curve C' and a map of pointed curves $C' \rightarrow C$, contracting precisely the components of C' , assigned by \mathcal{Z} , in a certain controlled way. In this paper we consider only one extremal assignment which associates to every stable curve all of its unmarked components (see [10, Ex. 1.12]), so when we say \mathcal{Z} -stable we always mean this particular extremal assignment.

We prove that the map ϕ_2 extends to a regular morphism of stacks

$$\phi_2 : \overline{\mathcal{M}}_{2,1}(\mathcal{Z}) \rightarrow \overline{\mathcal{U}}_{2,1}^{ns}(2)$$

contracting the closure of \mathcal{W} to one point (see Theorem 2.4.5). Furthermore, we identify the point $\phi_2(\mathcal{W})$ explicitly as a certain cuspidal curve C_0 (see Definition 2.2.2), and show that ϕ_2 induces an isomorphism of the complement of \mathcal{W} to the complement of $\phi_2(\mathcal{W})$.

We also prove that the natural rational map of the coarse moduli spaces $\overline{\mathcal{M}}_{2,1} \dashrightarrow \overline{\mathcal{U}}_{2,1}^{ns}(2)$ is a birational contraction with the exceptional divisors \overline{W} and Δ_1 (see Proposition 2.4.6). One can expect that the rational map $\overline{\mathcal{M}}_{g,1} \dashrightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g)$ is still a birational contraction for $g > 2$ (see Remark 2.4.7 for further discussion).

In addition, in Sec. 2.1 we obtain an isomorphism

$$\overline{\mathcal{U}}_{2,1}^{ns}(2) \simeq \mathbb{P}(2, 3, 4, 5, 6),$$

where the right-hand side is the weighted projective stack.

Conventions. In Sec. 2.1 we work over $\mathbb{Z}[1/6]$. Everywhere else we work over \mathbb{C} . By a curve we mean a connected reduced projective curve. By the genus of a curve we always mean arithmetic genus. For DM-stacks whose notation involves calligraphic letters \mathcal{M}, \mathcal{U} and \mathcal{W} , we denote their coarse moduli spaces by replacing these letters by M, U and W .

1. RATIONAL MAPS FOR₂ AND ϕ

1.1. Moduli spaces of curves with non-special divisors. We start by recalling some results from [8] about the stacks $\mathcal{U}_{g,n}^{ns}(\mathbf{a})$, where $\mathbf{a} = (a_1, \dots, a_n)$ and a_i are non-negative integers with $a_1 + \dots + a_n = g$. We denote by $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$ the \mathbb{G}_m^n -torsor over $\mathcal{U}_{g,n}^{ns}(\mathbf{a})$, corresponding to choices of nonzero tangent vectors at the marked points. It is proved in [8] that $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$ is an affine scheme of finite type. In this paper we only need the case when all a_i are positive, so we assume this is the case.

The key result we will use is that for each $i = 1, \dots, n$, and each $(C, p_1, \dots, p_n, v_1, \dots, v_n)$ in $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$ (where v_i is a nonzero tangent vector at p_i), there is a canonical formal parameter t_i on C at p_i , such that $\langle v_i, dt_i \rangle = 1$, which is defined as follows. Given a formal parameter t_i , for each $m > a_i$ there is unique, up to adding a constant, rational function $f_i[-m] \in H^0(C, \mathcal{O}(mp_i + \sum_{j \neq i} a_j p_j))$ with the Laurent expansion in t_i of the form

$$f_i[-m] = t_i^{-m} + \sum_{q \geq -a_i} \alpha_i[-m, q] t_i^q. \quad (1.1.1)$$

The canonical parameter is uniquely characterized by the condition that $\alpha_i[-m, -a_i] = 0$ for every $m > a_i$. Using these formal parameters we can consider for every pair (i, j) and $m > a_i$ the expansion of $f_i[-m]$ at p_j :

$$f_i[-m] = \sum_{q \geq -a_j} \alpha_{ij}[-m, q] t_i^q$$

(note that $\alpha_i[-m, q] = \alpha_{ii}[-m, q]$). Now we can view the coefficients $\alpha_{ij}[-m, q]$ as functions on $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$, where we fix the ambiguity in adding a constant to $f_i[-m]$ by requiring that $\alpha_i[-m, 0] = 0$. It follows from the results of [8] that these functions are all expressed in terms of a finite number of them, which gives a closed embedding of $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$ into an affine space.

The rescaling of the tangent vectors (v_i) defines an action of \mathbb{G}_m^n on $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$, so that the weight of the function $\alpha_{ij}[-m, q]$ is $m\mathbf{e}_i + q\mathbf{e}_j$, where (\mathbf{e}_i) is the standard basis in the character lattice of \mathbb{G}_m^n .

There is a special point in $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$ which is a unique point stable under the action of \mathbb{G}_m^n : it is the point where all the functions $\alpha_{ij}[-m, q]$ vanish, i.e., it corresponds to the origin in the ambient affine space. The underlying curve is the union of n rational cuspidal curves $C^{\text{cusp}}(a_i)$, glued transversally at the cusp. Here $C^{\text{cusp}}(a)$ is the projective curve with the affine part given by $\text{Spec}(k \cdot 1 + x^{a+1}k[x])$, and with one smooth point at infinity (see [8, Sec. 2.1]).

In [8] we also studied the GIT picture for the \mathbb{G}_m^n -action on $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$. In general we have stability conditions depending on a character χ of \mathbb{G}_m^n . In the case $n = 1$, i.e., for $\tilde{\mathcal{U}}_{g,1}^{ns}(g)$ there is a unique nonempty stability condition, so that the unique unstable point in $\tilde{\mathcal{U}}_{g,1}^{ns}(g)$ is the origin, i.e., the point corresponding to the curve $C^{\text{cusp}}(g)$. We denote this point by $[C^{\text{cusp}}(g)]$. Then the functions $\alpha_{ij}[-m, q]$ identify the corresponding GIT quotient stack,

$$\overline{\mathcal{U}}_{g,1}^{ns}(g) := (\tilde{\mathcal{U}}_{g,1}^{ns}(g) \setminus [C^{\text{cusp}}(g)]) / \mathbb{G}_m,$$

with a closed substack in the weighted projective stack.

For two collection of weights as above, \mathbf{a} and \mathbf{a}' , we denote by $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a}, \mathbf{a}')$ the interesection of the stacks $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$ and $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a}')$. In other words, we impose both conditions, $h^1(\sum a_i p_i) = 0$ and $h^1(\sum a'_i p_i) = 0$, on the marked points.

1.2. The forgetful map. The rational map (0.0.1) corresponds to a regular morphism

$$\text{for}_2 : \tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g, 0)) \rightarrow \tilde{\mathcal{U}}_{g,1}^{ns}(g), \quad (1.2.1)$$

which is given as the composition of the open embedding $\tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g, 0)) \hookrightarrow \tilde{\mathcal{U}}_{g,2}^{ns}(g, 0)$ followed by the forgetful map

$$\text{for}_2 : \tilde{\mathcal{U}}_{g,2}^{ns}(g, 0) \rightarrow \tilde{\mathcal{U}}_{g,1}^{ns}(g)$$

defined in [8, Thm. A]. The latter map sends (C, p_1, p_2, v_1, v_2) , with C irreducible, to (C, p_1, v_1) (if C is reducible then it gets replaced by a certain curve \overline{C} , such that $C \rightarrow \overline{C}$ is contraction of the component containing p_2).

Let $Z \subset \tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g, 0))$ be the closed subscheme given as the preimage of the origin under (1.2.1). Then there is a regular morphism

$$\tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g, 0)) \setminus Z \rightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g) \quad (1.2.2)$$

induced by (1.2.1). Note that Z consists of (C, p_1, p_2, v_1, v_2) such that (C, p_1) is the cuspidal curve $C^{\text{cusp}}(g)$ (with the marked point at infinity).

Let us denote by

$$\tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g+1, 0)) \subset \tilde{\mathcal{U}}_{g,2}^{ns}(g-1, 1)$$

the open subset given by the condition $h^1((g+1)p_1) = 0$. Let also

$$\widetilde{\mathcal{W}} \subset \tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g+1, 0))$$

denote the closed locus given by the condition $h^1(gp_1) \neq 0$, so that

$$\tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g, 0)) = \tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g+1, 0)) \setminus \widetilde{\mathcal{W}}.$$

Recall that we have sections $f_1[-m] \in H^0(C, \mathcal{O}(mp_1 + p_2))$, where C is the universal curve over $\tilde{\mathcal{U}}_{g,2}^{ns}(g-1, 1)$, for $m \geq g$, with expansions at p_1 of the form (1.1.1) (with $i = 1$) with $\alpha_1[-m, -g+1] = \alpha_1[-m, 0] = 0$.

Lemma 1.2.1. *Let us set $\alpha = \alpha_{12}[-g, -1]$, $\beta = \alpha_{12}[-g-1, -1]$. Then the open subset*

$$\tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g, 0)) \subset \tilde{\mathcal{U}}_{g,2}^{ns}(g-1, 1)$$

is given by the condition $\alpha \neq 0$. Similarly, the open subset

$$\tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g+1, 0)) \subset \tilde{\mathcal{U}}_{g,2}^{ns}(g-1, 1)$$

is the locus where either $\alpha \neq 0$ or $\beta \neq 0$.

Proof. Recall that the open subset $\tilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g, 0))$ is characterized by the condition $h^1(gp_1) = 0$. Since $h^1(gp_1 + p_2) = 0$, the long exact sequence of cohomology associated with the exact sequence of sheaves

$$0 \rightarrow \mathcal{O}(gp_1) \rightarrow \mathcal{O}(gp_1 + p_2) \rightarrow \mathcal{O}(p_2)/\mathcal{O} \rightarrow 0$$

shows that $h^1(gp_1) \neq 0$ precisely for those curves for which $f_1[-g]$ is regular at p_2 . But this is equivalent to the vanishing of α , since α is the coefficient of t_2^{-1} in the expansion of $f_1[-g]$ at p_2 .

The case of $\tilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g+1,0))$ is similar: now we consider the exact sequence

$$0 \rightarrow \mathcal{O}((g+1)p_1) \rightarrow \mathcal{O}((g+1)p_1 + p_2) \rightarrow \mathcal{O}(p_2)/\mathcal{O} \rightarrow 0$$

which shows that $h^1((g+1)p_1) \neq 0$ when both $f_1[-g]$ and $f_1[-g-1]$ are regular at p_2 , i.e., both α and β vanish. \square

The following Proposition is a crucial step in proving Theorem A.

Proposition 1.2.2. *The subset Z is closed in $\tilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g+1,0))$, and we have $Z \cap \tilde{\mathcal{W}} = \emptyset$. There exists a regular morphism*

$$\widetilde{\text{for}}_2 : \tilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g+1,0)) \setminus Z \rightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g), \quad (1.2.3)$$

extending the morphism (1.2.2) and sending $\tilde{\mathcal{W}}$ to a point. Furthermore, this point has no nontrivial automorphisms.

Proof. Let C' denote the universal curve over the open subset $\tilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g,0))$. To calculate explicitly the map (1.2.1), we need to find the sections $f[-m] \in H^0(C', \mathcal{O}(mp_1))$, for $m \geq g+1$, and a modified formal parameter u at p_1 , such that $f[-m]$ would have expansions of the form

$$f[-m] = u^{-m} + \alpha[-m, -g+1]u^{-g+1} + \alpha[-m, -g+2]u^{-g+2} + \dots, \quad (1.2.4)$$

where $\alpha[-m, q]$ are some rational expressions of the coordinates on $\tilde{\mathcal{U}}_{g,2}^{ns}(g-1,1)$ with only powers of α in the denominator.

As the first approximation let us set for $m \geq g+1$,

$$\tilde{f}[-m] = f_1[-m] - \frac{\alpha_{12}[-m, -1]}{\alpha} f_1[-g].$$

The constant is chosen so that the poles at p_2 cancel out, so we have $\tilde{f}[-m] \in H^0(\mathcal{O}(mp_1))$, while the expansion of $\tilde{f}[-m]$ at p_1 has form

$$\tilde{f}[-m] = t_1^{-m} - \frac{\alpha_{12}[-m, -1]}{\alpha} \cdot t_1^{-g} + \dots,$$

where t_1 is the canonical parameter at p_1 on C' .

Now we need to change the canonical parameter to $u = t_1 + c_1 t_1^2 + \dots$, and to add to each $\tilde{f}[-m]$ a linear combination of $\tilde{f}[-m']$ with $m' < m$, to get the expansions of the required form (1.2.4). We want to know only the highest order polar parts of the functions $\alpha[-m, q]$, i.e., those with the highest power of α (prescribed below) in the denominator, so we introduce the following filtration F_n on the space of formal Laurent series in t_1 with coefficients in $R = \mathcal{O}(\tilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g,0)))$. By definition, a Laurent series belongs to F_n if it can be written in the form $\sum_i a_i \alpha^{-i-n} t_1^i$, where each a_i extends to a regular function on $\tilde{\mathcal{U}}_{g,2}^{ns}(g-1,1)$.

It will be enough for us to keep track only of $f[-m] \bmod F_{m-1}$. It is easy to see that the change of variables $t_1 \mapsto t_1 + c_1 t_1^2 + c_2 t_1^3 + \dots$, where for each i , $\alpha^i c_i$ extends to a regular function on $\tilde{\mathcal{U}}_{g,2}^{ns}(g-1,1)$, preserves the filtration (F_n) . Since to go from t_1 to u we will only use the changes of variables of this form, it suffices for us to know that

$$\tilde{f}[-g-1] \equiv t_1^{-g-1} - \lambda t_1^{-g} \bmod F_g, \quad (1.2.5)$$

where $\lambda = \frac{\alpha_{12}[-g-1,-1]}{\alpha} = \frac{\beta}{\alpha}$, while

$$\tilde{f}[-m] \equiv t_1^{-m} \bmod F_{m-1} \quad \text{for } m > g+1. \quad (1.2.6)$$

We claim that there exist rational constants $(r_{m,j})$, $1 \leq j < m-g$, and (r_i) , $i \geq 1$, such that

$$f[-m] \equiv \tilde{f}[-m] + \sum_{1 \leq j < m-g} r_{m,j} \lambda^j \tilde{f}[-m+j] \bmod F_{m-1}, \quad (1.2.7)$$

for each $m \geq g+1$, and

$$t_1 \equiv u + r_1 \lambda u^2 + r_2 \lambda^2 u^3 + \dots \bmod F_{-2}. \quad (1.2.8)$$

Namely, we prove by induction on $n \geq 1$ that (1.2.7) holds for all m with $m \leq g+n$, and that the required relation between t_1 and u holds modulo $t_1^n R[[t_1]] + F_{-2}$.

Let us recall the recursive construction of $(f[-g-n])$ and of formal parameters u_n such that $u_n \equiv u \bmod t_1^{n+1} R[[t_1]]$, where u is the canonical parameter (cf. [4, Lem. 4.1.3]). For $n=1$ we have $f[-g-1] = \tilde{f}[-g-1]$ and $u_1 = t_1$. Assume $f[-g-n']$ are already defined for $n' < n$ and $u_{n-1} \equiv u \bmod t_1^n R[[t_1]]$ is known, so that

$$\begin{aligned} f[-g-n'] &\equiv u_{n-1}^{-g-n'} \bmod t_1^{-g+1} R[[t_1]] \quad \text{for } n' < n-1, \text{ while} \\ f[-g-n+1] &\equiv u_{n-1}^{-g-n+1} + c \cdot u_{n-1}^{-g} \bmod t_1^{-g+1} R[[t_1]]. \end{aligned} \quad (1.2.9)$$

Then we set $u_n = u_{n-1} + \frac{c}{g+n-1} u_{n-1}^n$, the expansion of $f[-g-n+1]$ in u_n will take form

$$f[-g-n+1] \equiv u_n^{-g-n+1} \bmod t_1^{-g+1} R[[t_1]],$$

and the expansions of $f[-g-n']$ for all $n' < n-1$ in u_n will still have the correct form. Now, if the expansion of $\tilde{f}[-g-n]$ in u_n has form

$$\tilde{f}[-g-n] = u_n^{-g-n} + p_1 u_n^{-g-n+1} + \dots + p_{n-1} u_n^{-g-1} + \dots, \quad (1.2.10)$$

then we set

$$f[-g-n] = \tilde{f}[-g-n] - p_1 f[-g-n+1] - \dots - p_{n-1} f[-g-1]. \quad (1.2.11)$$

The induction assumption implies that the function c in (1.2.9) has the leading polar term $r \lambda^{n-1}$ for some $r \in \mathbb{Q}$, so the change of variables from u_{n-1} to u_n is of the right form, as discussed above. It follows that

$$t_1 \equiv u_n + s_1 \lambda u_n^2 + \dots + s_{n-1} \lambda^{n-1} u_n^n \bmod t_1^{n+1} R[[t_1]] + F_{-2}$$

for some $s_i \in \mathbb{Q}$. Now from (1.2.6) we get that

$$\tilde{f}[-g-n] = (u_n + s_1 \lambda u_n^2 + \dots + s_{n-1} \lambda^{n-1} u_n^n)^{-g-n} \bmod t_1^{-g} R[[t_1]] + F_{g+n-1}.$$

This implies that for $i = 1, \dots, n = 1$, the leading polar term of the coefficient p_i in the expansion (1.2.10) is of the form $a_i \lambda^i$, for $a_i \in \mathbb{Q}$. Now (1.2.11) shows that (1.2.7) holds for $m = g + n$. This finishes the proof of our claim.

Now combining (1.2.5)–(1.2.8), we get that for each $m \geq g + 1$ the expansion of $f[-m]$ in the canonical parameter u has form

$$f[-m] \equiv u^{-m} + \sum_{j \geq 1} s_{m,j} \lambda^{m-g+j} u^{-g+j} \bmod F_{m-1},$$

for some rational constants $(s_{m,j})$. In other words, the functions $\alpha[-m, -g + j] \in R$, defining the map (1.2.1), have form

$$\alpha[-m, -g + j] = s_{m,j} \lambda^{m-g+j} + \dots$$

where the omitted terms have smaller powers of α in the denominator.

Finally, we need to know that not all $(s_{m,j})$ are zero, so let us compute $s_{-g-1, -g+1}$ and $s_{g-1, -g+2}$ following the above procedure (we will need to look at two coordinates to prove that the point, which is the image of \mathcal{W} , has no nontrivial automorphisms). Due to (1.2.5), the first change of variables is

$$t_1 = u_2 - \frac{\lambda}{g+1} u_2^2 \bmod F_{-2}.$$

Then we get expansions

$$\begin{aligned} f[-g-1] &= \tilde{f}[-g-1] \equiv u_2^{-g-1} + \frac{2-g}{2(g+1)} \lambda^2 u_2^{-g+1} + \\ &\quad \frac{-g^2+g+3}{3(g+1)^2} \lambda^3 u_2^{-g+2} \bmod u_2^{-g+3} R[[u_2]] + F_g, \\ \tilde{f}[-g-2] &\equiv u_2^{-g-2} + \frac{g+2}{g+1} \lambda u_2^{-g-1} + \frac{(g+2)(g+3)}{2(g+1)^2} \lambda^2 u_2^{-g} \bmod u_2^{-g+1} R[[u_2]] + F_{g+1}, \\ \tilde{f}[-g-3] &\equiv u_2^{-g-3} + \frac{g+3}{g+1} \lambda u_2^{-g-2} + \frac{(g+3)(g+4)}{2(g+1)^2} \lambda^2 u_2^{-g-1} + \\ &\quad \frac{(g+3)(g+4)(g+5)}{6(g+1)^3} \lambda^3 u_2^{-g} \bmod u_2^{-g+1} R[[u_2]] + F_{g+2}. \end{aligned}$$

Hence, the coefficient of u_2^{-g} in $f[-g-2] \bmod F_{g+1}$ (which is the same as in $\tilde{f}[-g-2] \bmod F_{g+1}$) is $\frac{(g+2)(g+3)}{2(g+1)^2} \lambda^2$. Thus, the second change of variables (defined so that the coefficient of u_3^{-g} in $f[-g-2]$ is zero) is

$$u_2 = u_3 + \frac{(g+2)(g+3)}{2(g+2)(g+1)^2} \lambda^2 u_3^3 \bmod F_{-2},$$

and we get the expansion

$$f[-g-1] = u_3^{-g-1} - \frac{2g+1}{2(g+1)} \lambda^2 u_3^{-g+1} + \frac{-g^2+g+3}{3(g+1)^2} \lambda^3 u_3^{-g+2} \bmod u_3^{-g+3} R[[u_3]] + F_g,$$

which shows that

$$s_{g+1,1} = -\frac{2g+1}{2(g+1)}.$$

Also, we see that the coefficient of u_3^{-g} in the expansion of $\tilde{f}[-g-3] \bmod F_{g+2}$ is equal to $-\frac{(g+3)(g^2+3g-1)}{3(g+1)^3}\lambda^3$. This dictates that the next change of variables is

$$u_3 = u_4 - \frac{(g^2+3g-1)}{3(g+1)^3}\lambda^3 u_4 \bmod F_{-2}.$$

Finally, we get that the coefficient of u_4^{-g+2} in the expansion of $f[-g-1] \bmod F_g$ is equal to

$$\frac{-g^2+g+3}{3(g+1)^2}\lambda^3 + \frac{(g^2+3g-1)}{3(g+1)^2}\lambda^3 = \frac{4g+2}{3(g+1)^2}\lambda^3,$$

and hence,

$$s_{g+1,2} = \frac{4g+2}{3(g+1)^2}.$$

Now let us consider the modified map

$$\alpha \cdot \widetilde{\text{for}}_2 : \widetilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g,0)) \rightarrow \widetilde{\mathcal{U}}_{g,1}^{ns}(g) : x \mapsto \alpha(x) \cdot \widetilde{\text{for}}_2(x)$$

Since the weight of $\alpha[-m, -g+j]$ is $m-g+j$, the modified map sends x to the point in $\widetilde{\mathcal{U}}_{g,1}^{ns}(g)$ with coordinates

$$\alpha(x)^{m-g+j} \alpha[-m, -g+j](x) = s_{m,j} \beta(x)^{m-g+j} + \alpha(x) \cdot f_{m,j}(x), \quad (1.2.12)$$

where $f_{m,j}$ are regular functions on $\widetilde{\mathcal{U}}_{g,2}^{ns}(g-1,1)$. In particular, $\alpha \cdot \widetilde{\text{for}}_2$ can be viewed as a regular map from $\widetilde{\mathcal{U}}_{g,2}^{ns}(g-1,1)$.

Recall that by Lemma 1.2.1, the open subset $\widetilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g+1,0))$ is the locus where either $\alpha \neq 0$ or $\beta \neq 0$, and the locus $\widetilde{\mathcal{W}}$ is given by $\alpha = 0$. Thus, (1.2.12) gives for $x \in \widetilde{\mathcal{W}}$:

$$\alpha \cdot \widetilde{\text{for}}_2(x) = (s_{m,j} \beta(x)^{m-g+j}) = \beta(x) \cdot (s_{m,j}).$$

Furthermore, as we have seen above, the constants $s_{-g-1,1}$ and $s_{-g-1,2}$ are nonzero, so the corresponding coordinates in the above expression are also nonzero. Note also that the corresponding point of $\widetilde{\mathcal{U}}_{g,1}^{ns}(g)$ is equal to $(s_{m,j})$, so it does not depend on x .

Denoting by $U_{\beta \neq 0} \subset \widetilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g+1,0))$ the open subset where $\beta \neq 0$, we get

$$(\alpha \cdot \widetilde{\text{for}}_2)^{-1}(0) \cap U_{\beta \neq 0} = Z \cap U_{\beta \neq 0},$$

and so $Z \cap U_{\beta \neq 0}$ is closed in $U_{\beta \neq 0}$. Since, Z is closed in the open subset $\alpha \neq 0$, we derive that Z is closed in $\widetilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g+1,0))$.

We have a covering of $\widetilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g+1,0)) \setminus Z$ by two open subsets: $\widetilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g,0)) \setminus Z$ and $U_{\beta \neq 0}$. The required regular morphism (1.2.3) to $\widetilde{\mathcal{U}}_{g,1}^{ns}(g)$ is induced by $\widetilde{\text{for}}_2$ on $\widetilde{\mathcal{U}}_{g,2}^{ns}((g-1,1), (g,0)) \setminus Z$ and by $\alpha \cdot \widetilde{\text{for}}_2$ on $U_{\beta \neq 0}$. As we have seen above, this morphism sends $\widetilde{\mathcal{W}} \subset U_{\beta \neq 0}$ to the point $(s_{m,j})$ of the weighted projective stack with two

nonzero homogeneous coordinates, of weights 2 and 3. Hence, this point does not have nontrivial automorphisms. \square

1.3. Proof of Theorem A. It is well known that $\mathcal{W} \subset \mathcal{M}_{g,1}$ is an irreducible divisor (see [1, 2]). Now let $U \subset \mathcal{M}_{g,1}$ be the open substack of (C, p) satisfying

$$h^1((g+1)p) = 0, \quad h^0((g-1)p) = 1.$$

Note also that we have an inclusion

$$\mathcal{M}_{g,1} \setminus \mathcal{W} \subset U$$

since the condition $h^0(gp) = 1$ implies that $h^1((g+1)p) = 0$ and $h^0((g-1)p) = 1$. Furthermore, the complement to U is a proper closed subset in \mathcal{W} , so it has codimension ≥ 2 in $\mathcal{M}_{g,1}$. In particular, $U \cap \mathcal{W}$ is dense in \mathcal{W} .

Note that we have a natural open inclusion

$$\mathcal{M}_{g,1} \setminus \mathcal{W} \hookrightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g). \quad (1.3.1)$$

Indeed, the only unstable point in $\mathcal{U}_{g,1}^{ns}(g)$ corresponds to the singular curve $C^{\text{cusp}}(g)$. We are going to show that the above morphism extends to a regular morphism

$$U \rightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g),$$

such that $U \cap \mathcal{W}$ is mapped to a point.

Recall that by Proposition 1.2.2, we have a regular morphism

$$\widetilde{\text{for}}_2 : \widetilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g+1, 0)) \setminus Z \rightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g),$$

sending $\widetilde{\mathcal{W}}$ to a point. Let $V \subset \widetilde{\mathcal{U}}_{g,2}^{ns}((g-1, 1), (g+1, 0))$ be the open subset corresponding to smooth curves. Then $V \cap Z = \emptyset$ because for points of Z the underlying curve is singular. Thus, the above morphism induces a regular morphism

$$\widetilde{\phi} : V \rightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g), \quad (1.3.2)$$

mapping $\widetilde{\mathcal{W}} \cap V$ to a point.

Now we claim that the natural projection $V \rightarrow \mathcal{M}_{g,1}$ induces a smooth surjective morphism $V \rightarrow U$. Indeed, if $h^0((g-1)p_1 + p_2) = 1$ then $h^0((g-1)p_1) = 1$, so this projection factors through U . Conversely, if for $(C, p_1) \in \mathcal{M}_{g,1}$ one has $h^0((g-1)p_1) = 1$ then for generic p_2 we will have $h^0((g-1)p_1 + p_2) = 1$, hence the map $V \rightarrow U$ is surjective. It is smooth since V is a \mathbb{G}_m^2 -torsor over an open substack of a universal curve over U .

It remains to prove that the morphism (1.3.2) factors through a morphism $\phi : U \rightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g)$ (it will then map $\mathcal{W} \cap U$ to a point, since (1.3.2) sends $\widetilde{\mathcal{W}} \cap V$ to a point). Indeed, this is true if we restrict to the open subset $\mathcal{M}_{g,1} \setminus \mathcal{W}$, by the construction. Now let us set $T := V \times_U V$ and consider two morphisms

$$f_1 = \widetilde{\phi} \circ \pi_1, f_2 = \widetilde{\phi} \circ \pi_2 : T \rightarrow \overline{\mathcal{U}}_{g,1}^{ns}(g),$$

where π_1 and π_2 are two projections to V . We know that these two maps agree on the open subset $\pi^{-1}(\mathcal{M}_{g,1} \setminus \mathcal{W})$, where π is the projection $T = V \times_U V \rightarrow U$.

Note that the scheme T parametrizes data $(C, p_1, p_2, p'_2, v_1, v_2, v'_2)$ such that $h^0((g-1)p_1 + p_2) = h^0((g-1)p_1 + p'_2) = 1$ and $h^1((g+1)p_1) = 0$ (and C smooth, $p_1 \neq p_2$,

$p_1 \neq p'_2$). Thus, it is an open subset in a \mathbb{G}_m^3 -torsor over the universal curve over $\mathcal{M}_{g,2}$ (via the projection to (C, p_1, p_2, p'_2)), in particular, T is smooth and irreducible.

Let us consider the cartesian diagram

$$\begin{array}{ccc} T' & \xrightarrow{\quad} & \overline{\mathcal{U}}_{g,1}^{ns}(g) \\ \rho \downarrow & & \Delta \downarrow \\ T & \xrightarrow{(f_1, f_2)} & \overline{\mathcal{U}}_{g,1}^{ns}(g) \times \overline{\mathcal{U}}_{g,1}^{ns}(g) \end{array}$$

Since the stack $\overline{\mathcal{U}}_{g,1}^{ns}(g)$ is separated, the vertical arrows are finite morphisms. Finally, we observe that a generic pointed curve (C, p) in $\mathcal{M}_{g,1}$ does not have nontrivial automorphisms (note that in the case $g = 2$ this is true since we can take p not to be a Weierstrass point). Hence, the preimages of points with trivial automorphisms in $\overline{\mathcal{U}}_{g,1}^{ns}(g)$ under f_1 and f_2 are nonempty open subsets in T . Since f_1 and f_2 agree on a nonempty open subset, we deduce that there exists a nonempty open subset $W \subset T$ such that $\rho^{-1}(W) \rightarrow W$ is an isomorphism. Let $T'' \subset T'$ be an irreducible component of T' , containing $\rho^{-1}(W)$, with reduced scheme structure. Then $\rho|_{T''} : T'' \rightarrow T$ is a finite birational morphism. Since T is smooth, we deduce that $\rho|_{T''}$ is an isomorphism. Hence, ρ admits a section, and so we have $f_1 = f_2$, which means that the map (1.3.2) descends to a morphism from U . \square

2. CURVES OF GENUS 2

2.1. Explicit identification of $\tilde{\mathcal{U}}_{2,1}^{ns}(2)$.

Proposition 2.1.1. *Let us work over $\mathbb{Z}[1/6]$. One has an isomorphism of the moduli scheme $\tilde{\mathcal{U}}_{2,1}^{ns}(2)$ with the affine space \mathbb{A}^5 with coordinates $q_1, q_{2,0}, q_{2,1}, q_{3,0}, q_{3,1}$, so that the affine universal curve $C \setminus \{p\}$ is given by the following equations in the independent variables f, h, k :*

$$\begin{aligned} h^2 &= fk + q_1h + 2q_1^2 + f(q_{2,0} + q_{2,1}f), \\ hk &= f(q_{3,0} + q_{3,1}f + f^2) - q_1k + (q_{2,0} + q_{2,1}f)h + q_1(q_{2,0} + q_{2,1}f), \\ k^2 &= (q_{3,0} + q_{3,1}f + f^2)h + (q_{2,0} + q_{2,1}f)^2 - 2q_1(q_{3,0} + q_{3,1}f + f^2). \end{aligned} \quad (2.1.1)$$

The weights of the \mathbb{G}_m -action are:

$$\deg(q_{2,1}) = 2, \quad \deg(q_{3,1}) = 3, \quad \deg(q_1) = 4, \quad \deg(q_{2,0}) = 5, \quad \deg(q_{3,0}) = 6.$$

Hence, we get the identification of $\overline{\mathcal{U}}_{2,1}^{ns}(2)$ with the weighted projective stack $\mathbb{P}(2, 3, 4, 5, 6)$.

Proof. This is proved using the same method as in [7, Thm. A] and [8, Thm. A]. Let (C, p, v) be a point in $\tilde{\mathcal{U}}_{2,1}^{ns}(2)$. Since $h^1(2p) = 0$, we have $h^0(np) = n - 1$ for $n \geq 2$. Let t be a formal parameter at t compatible with the given tangent vector. We can find the elements $f \in H^0(C, \mathcal{O}(3p))$, $h \in H^0(C, \mathcal{O}(4p))$ and $k \in H^0(C, \mathcal{O}(5p))$ with the Laurent expansions

$$f = \frac{1}{t^3} + \dots, \quad h = \frac{1}{t^4} + \dots, \quad k = \frac{1}{t^5} + \dots,$$

where the omitted terms have poles of smaller order. Then the elements

$$f^n, f^n h, f^n k, \quad \text{for } n \geq 0, \quad (2.1.2)$$

form a linear basis on $H^0(C \setminus \{p\}, \mathcal{O})$, so we can express h^2 , hk and k^2 as their linear combinations. Taking into account the above Laurent expansion, we get relations of the form

$$\begin{aligned} h^2 &= p_1(f)k + q_1(f)h + c_1(f), \\ hk &= p_2(f)k + q_2(f)h + c_2(f), \\ k^2 &= p_3(f)k + q_3(f)h + c_3(f), \end{aligned} \quad (2.1.3)$$

where p_i , q_i , c_i are polynomials in f with the following restrictions:

$$\begin{aligned} \deg p_1 &= 1, \deg p_2 \leq 1, \deg p_3 \leq 1, \deg q_1 \leq 1, \deg q_2 \leq 1, \deg q_3 = 2, \\ \deg c_1 &\leq 2, \deg c_2 = 3, \deg c_3 \leq 3, \end{aligned}$$

and the polynomials p_1 , q_3 and c_2 are monic. Note that f is defined up to adding a constant, while h and k are defined up to the transformation

$$(h, k) \mapsto (\tilde{h} = h + A(f), \tilde{k} = k + Bh + C(f)),$$

where A and C are linear polynomials in f and B is a constant. It is easy to check that we can fix the ambiguity in the choice of h and k by requiring that $p_3 = 0$ and $p_2 = -q_1$ is a constant, i.e., does not have a linear term in f . More precisely, we should set

$$A = -\frac{q_1 + p_2}{3}, \quad B = \frac{1}{3}(q'_1 - 2p'_2), \quad C = -\frac{p_3}{2} - \frac{B^2 p_1}{2} - Bp_2 \quad (2.1.4)$$

(here q'_1 and p'_2 are derivatives of the linear polynomials q_1 and p_2). Note that here we use our assumption that 6 is invertible. Finally, we can fix the ambiguity in the choice of f by requiring that $p_1(f) = f$.

Now the fact that the elements (2.1.2) form a basis of $H^0(C \setminus \{p\}, \mathcal{O})$ is equivalent to the condition that the relations (2.1.3) form a Gröbner basis in the ideal they generate (with respect to the degree reverse lexicographical order such that $f < h < k$, $\deg(f) = 3$, $\deg(h) = 4$, $\deg(k) = 5$). Applying the Buchberger's Criterion (see [3, Thm. 15.8]) we compute that this condition is equivalent to the following expressions of c_1, c_2, c_3 in terms of the other variables (where in the second expression in each line we take into account the normalization $p_3 = 0$, $p_2 = -q_1$):

$$\begin{aligned} c_1 &= p_2^2 + p_1 q_2 - q_1 p_2 = 2q_1^2 + p_1 q_2, \\ c_2 &= p_1 q_3 - p_2 q_2 = p_1 q_3 + q_1 q_2, \\ c_3 &= q_2^2 + p_2 q_3 - q_1 q_3 = q_2^2 - 2q_1 q_3. \end{aligned}$$

Thus, if we set

$$q_2 = q_{2,0} + q_{2,1}f, \quad q_3 = q_{3,0} + q_{3,1}f + f^2,$$

then we see that the constants $(q_1, q_{2,0}, q_{2,1}, q_{3,0}, q_{3,1})$ determine the curve (C, p) . The above process can be run in families and can be reversed (see the proofs of [7, Thm. A] and [8, Thm. A]), so this gives the required identification of our moduli space with \mathbb{A}^5 . \square

2.2. Special cuspidal curve C_0 . Let C_0 denote the curve obtained from \mathbb{P}^1 by pinching the point 0 into a genus 2 cuspidal singular point, so that a regular function f near 0 descends to C_0 if and only if the expansion of f in the standard parameter t has form

$$f \equiv c_0 + c_2 \cdot t^2 \pmod{t^4}. \quad (2.2.1)$$

Note that this condition depends on coordinates, i.e., the point $\infty \in C_0$ plays a special role. For example, the standard \mathbb{G}_m -action on \mathbb{P}^1 , preserving 0 and ∞ , descends to a \mathbb{G}_m -action on C_0 . Also, note that $C_0 \setminus \{\infty\} = \text{Spec}(\mathbb{C}[t^2, t^5])$.

The next Lemma shows that if we equip C_0 with a smooth marked point $p \neq \infty$ then we get a point of $\overline{\mathcal{U}}_{2,1}^{ns}(2)$.

Lemma 2.2.1. *Let $p \in C_0 \setminus \{0, \infty\}$. Then $h^0(C_0, \mathcal{O}(2p)) = 1$. On the other hand, for $p = \infty$ we have $h^0(C_0, \mathcal{O}(2p)) = 2$.*

Proof. In the case $p \neq 0, \infty$ we can assume that $t(p) = 1$. Then $\mathcal{O}_{\mathbb{P}^1}(2p)$ is spanned by 1, $\frac{1}{1-t}$ and $\frac{1}{(1-t)^2}$. Looking at the expansions at $t = 0$ we see that the only sections of $\mathcal{O}_{\mathbb{P}^1}(2p)$ satisfying (2.2.1) are constants.

In the case $p = \infty$ the functions $(1, t^2)$ give a basis of $H^0(C_0, \mathcal{O}(2p))$. \square

Definition 2.2.2. We denote by $[C_0]$ the point of $\overline{\mathcal{U}}_{2,1}^{ns}(2)$ corresponding to (C_0, p) , where $p \neq 0, \infty$.

2.3. Classification of singular irreducible curves of genus 2. Let C be an irreducible curve of genus 2, and let $\rho : \tilde{C} \rightarrow C$ be the normalization. If C is singular then the genus of \tilde{C} is either 1 or 0.

If the genus of \tilde{C} is 1 then $\text{coker}(\mathcal{O}_C \rightarrow \rho_* \mathcal{O}_{\tilde{C}})$ has length 1, so it is supported at one singular point $q \in C$. If $\rho^{-1}(q)$ contains two distinct points $q_1, q_2 \in \tilde{C}$ then ρ factors through a morphism $C' \rightarrow C$, where C' is the nodal curve obtained by gluing q_1 and q_2 on \tilde{C} . Since C' has genus 2 we should have $C \simeq C'$. If $\rho^{-1}(q)$ is one point on \tilde{C} then it is easy to see that C has a simple cusp at q .

In the remaining case when $\tilde{C} = \mathbb{P}^1$ we have more possibilities. The length of the sheaf $\mathcal{F} := \text{coker}(\mathcal{O}_C \rightarrow \rho_* \mathcal{O}_{\tilde{C}})$ is now 2, so the support of \mathcal{F} can consist of ≤ 2 points.

Case I: support of \mathcal{F} consists of two distinct points q_1, q_2 . We have the following subcases.

Case Ia: $|\rho^{-1}(q_1)| > 1$ and $|\rho^{-1}(q_2)| > 1$. In this case the map ρ factors through the nodal curve C' obtained by gluing two pairs of distinct points in \mathbb{P}^1 . Since the genus of C' is 2, we should have $C \simeq C'$.

Case Ib: $|\rho^{-1}(q_1)| = 1$ and $|\rho^{-1}(q_2)| > 1$. In this case ρ factors through the curve C' obtained by gluing two pairs of distinct point in \mathbb{P}^1 and pinching one extra point to a simple cusp. Again, we have that the genus of C' is 2, so $C \simeq C'$.

Case Ic: $|\rho^{-1}(q_1)| = |\rho^{-1}(q_2)| = 1$. In this case C is obtained by pinching two points of \mathbb{P}^1 into simple cusps.

Case II: \mathcal{F} is supported at one point q .

Case IIa: $|\rho^{-1}(q)| > 2$. In this case ρ factors through the curve C' obtained by gluing transversally 3 points on \mathbb{P}^1 into a single point (with the coordinate cross singularity). Since the genus of C' is 2, we get $C \simeq C'$.

Case IIb: $|\rho^{-1}(q)| = 2$. Let $\rho^{-1}(q) = \{q_1, q_2\}$. Let t be a generator of the maximal ideal $\mathfrak{m}_q \subset \mathcal{O}_{C,q}$. Assume first that $t \in \mathfrak{m}_{q_1}^2$. Then ρ factors through the curve C' obtained from \mathbb{P}^1 by first pinching q_1 into a simple cusp and then gluing it transversally with the point q_2 . Since C' has genus 2, we have $C \simeq C'$. On the other hand, if t maps to a generator of \mathfrak{m}_{q_i} for $i = 1, 2$, then ρ factors through the curve C' obtained from \mathbb{P}^1 by gluing q_1 and q_2 into a tacnode singularity. Since such C' has genus 2, we have $C \simeq C'$.

Case IIc: $|\rho^{-1}(q)| = 1$. In this case we can identify C with \mathbb{P}^1 as a topological space, so that \mathcal{O}_C is a subsheaf of $\mathcal{O}_{\mathbb{P}^1}$, which differs from it only at one point q , so that $\mathfrak{m}_{C,q} \subset \mathfrak{m}_{\mathbb{P}^1,q}^2$ is an embedding of codimension 1. We claim that there are two curves of this type, up to an isomorphism. If $\mathfrak{m}_{C,q} \subset \mathfrak{m}_{\mathbb{P}^1,q}^3$ then $\mathfrak{m}_{C,q} = \mathfrak{m}_{\mathbb{P}^1,q}^3$ and $C = C^{\text{cusp}}(2)$ (see Sec. 1.1). Now assume that $\mathfrak{m}_{C,q} \not\subset \mathfrak{m}_{\mathbb{P}^1,q}^3$. Let t be a formal parameter near q on \mathbb{P}^1 . Then $\hat{\mathfrak{m}}_{C,q}$ is a (non-unital) subalgebra in $t^2\mathbb{C}[[t]]$ of codimension 1, and there exists an element $f \in \hat{\mathfrak{m}}_{C,q}$ such that $f \equiv t^2 \pmod{t^3\mathbb{C}[[t]]}$. Changing the formal parameter we can assume that $f = t^2$. There could not be an element $h \in \hat{\mathfrak{m}}_{C,q}$ such that $h \equiv t^3 \pmod{t^4\mathbb{C}[[t]]}$, since then we would have $\hat{\mathfrak{m}}_{C,q} = t^2\mathbb{C}[[t]]$. Therefore,

$$\hat{\mathfrak{m}}_{C,q} = \mathbb{C} \cdot t^2 + t^4\mathbb{C}[[t]].$$

Note that the subspace in the right-hand side depends only on $t \pmod{t^3\mathbb{C}[[t]]}$. Now we observe that any formal parameter at q , modulo $\mathfrak{m}_{\mathbb{P}^1,q}^3$, can be obtained from a unique regular function on $\mathbb{P}^1 \setminus \{p\}$, for some $p \neq q$. Using automorphisms of \mathbb{P}^1 we can make $q = 0$, $p = \infty$, so that C is the curve C_0 defined before.

2.4. Comparison of stabilities for irreducible curves of genus 2.

Proposition 2.4.1. *Let C be an irreducible curve of genus 2, and let p be a smooth point. Then (C, p) is \mathcal{Z} -stable if and only if C is not of type IIc.*

Proof. It is easy to see that a curve C of type IIc is not \mathcal{Z} -stable. Indeed, if there is a contracting map $C' \rightarrow C$ then C' would have a rational component with only two distinguished points, so it could not be stable. Assume now that C is not of type IIc. If (C, p) is nodal then it is stable (since C is irreducible), hence it is \mathcal{Z} -stable.

Next, if C is obtained by pinching a point on an irreducible nodal curve E of genus 1 into a cusp, then there is a contraction $f : E \cup E' \rightarrow C$, where $E \cup E'$ is the stable curve with E and E' glued nodally at one point. Here the marked point is placed on E and $f(E')$ is the cusp on C . This shows that (C, p) is \mathcal{Z} -stable. Similarly, if C is a rational curve with two cusps then there is a contraction to C from \mathbb{P}^1 with two elliptic tails (that get contracted into cusps).

There remains two cases for C : IIa and IIb. In the case IIa we have a contraction to C from the union of two \mathbb{P}^1 's, joined nodally at 2 points. In the case IIb there is a contraction to C from the curve with an elliptic bridge. In other words, we consider the union $\mathbb{P}^1 \cup E$, where E is an elliptic curve, \mathbb{P}^1 and E are joined nodally at 2 points, so that there are no marked points on E . It is known (see [10, Ex. 2.5]) that there exists a contraction $\mathbb{P}^1 \cup E \rightarrow C$, mapping E to the singular point, for both types of curves occurring in the case IIb. \square

Corollary 2.4.2. *The stack $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ is smooth and irreducible.*

Proof. The possible singular points that can appear in \mathcal{Z} -stable curves of genus 2, other than nodes, are: a simple cusp, a tacnode, and a coordinate cross in 3-space. All of these have smooth versal deformation spaces and are smoothable, hence the assertion (see [11, Lem. 2.1]). \square

Using the classification from Sec. 2.3 we easily get the following codimension estimate.

Lemma 2.4.3. *Away from a closed subset of codimension ≥ 2 , for every point (C, p) in $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ (resp., $\overline{\mathcal{U}}_{2,1}^{ns}(2)$), C is either smooth, or a nodal curve with the normalization of genus 1.*

Proof. Both $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ and $\overline{\mathcal{U}}_{2,1}^{ns}(2)$ are irreducible of dimension 4. Now we just go through the strata described in Sec. 2.3 and see that they all have dimension ≤ 2 , except when C is either smooth or nodal with the normalization of genus 1. \square

We need one more simple observation.

Lemma 2.4.4. *Let C be an irreducible curve of genus 2, and let $p \in C$ be a smooth point. Then $h^0(p) = 1$ and $h^1(3p) = 0$.*

Proof. First, if $h^0(p) = 2$ then we would get a degree 1 regular map $C \rightarrow \mathbb{P}^1$. Composing it with the normalization map $\tilde{C} \rightarrow C$, we get that the normalization map is the inverse map $\mathbb{P}^1 \rightarrow C$, which is impossible. Hence, $h^0(p) = 1$.

If $h^1(2p) = 0$ then we also have $h^1(3p) = 0$, so it is enough to consider the case $h^1(2p) \neq 0$, i.e., $h^0(2p) = 2$. Suppose that $h^0(3p) = 3$. Then we can choose $f \in H^0(C, \mathcal{O}(2p))$ and $h \in H^0(C, \mathcal{O}(3p))$ with the Laurent expansions $f = \frac{1}{t^2} + \dots$, $h = \frac{1}{t^3} + \dots$ at p (for some formal parameter t at p). Furthermore, there is a canonical choice of f and h , such that the relation

$$h^2 = f^3 + af + b$$

holds for some constants a and b . Then the algebra $\mathcal{O}(C \setminus \{p\})$ has the linear basis (f^n) , (hf^n) , and is isomorphic to the algebra $A = \mathbb{C}[h, f]/(h^2 - f^3 - af - b)$. Since C is irreducible, it is isomorphic to Proj of the Rees algebra of A , which is a plane cubic, so we get that the arithmetic genus of C is equal to 1, which is a contradiction. This shows that $h^0(3p) = 2$, i.e., $h^1(3p) = 0$. \square

Theorem 2.4.5. *Let $\overline{\mathcal{W}} \subset \overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ be the closure of the Weierstrass locus $\mathcal{W} \subset \mathcal{M}_{2,1}$. Then $\overline{\mathcal{W}}$ coincides with the locus where $h^1(2p) \neq 0$. There is a regular morphism*

$$\phi_2 : \overline{\mathcal{M}}_{2,1}(\mathcal{Z}) \rightarrow \overline{\mathcal{U}}_{2,1}^{ns}(2),$$

such that $\phi_2(\overline{\mathcal{W}}) = [C_0]$ and ϕ_2 induces an isomorphism

$$\overline{\mathcal{M}}_{2,1}(\mathcal{Z}) \setminus \overline{\mathcal{W}} \xrightarrow{\sim} \overline{\mathcal{U}}_{2,1}^{ns}(2) \setminus [C_0].$$

Proof. First, we observe that every irreducible component of the locus $h^1(2p) \neq 0$ has codimension 1 in $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ (recall that the latter stack is smooth and irreducible by Corollary 2.4.2). By Lemma 2.4.3, to see that this locus coincides with $\overline{\mathcal{W}}$, it is enough to see that the locus of (C, p) , such that C is nodal with normalization E of genus 1 and $h^1(2p) \neq 0$ has dimension 2 (and hence has codimension 2 in $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$). But if C is obtained from E

by identifying points $q_1 \neq q_2$ then the condition that $h^0(2p) = 2$ implies the existence of a rational function on E with pole of order 2 at p and vanishing at both q_1 and q_2 . In other words, we should have a linear equivalence $2p \sim q_1 + q_2$. Thus, we have a finite number of choices for each (E, q_1, q_2) , so the dimension is 2.

Next, let us denote by

$$V^{\mathcal{Z}} \subset \tilde{\mathcal{U}}_{2,2}^{ns}(1, 1)$$

the open substack consisting of (C, p_1, p_2) such that (C, p_1) is \mathcal{Z} -stable (in particular, C is irreducible). Lemma 2.4.4 shows that every (C, p) in $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ satisfies $h^0(p) = 1$ and $h^1(3p) = 0$. This implies that

$$V^{\mathcal{Z}} \subset \tilde{\mathcal{U}}_{2,2}^{ns}((1, 1), (3, 0))$$

and the projection $V^{\mathcal{Z}} \rightarrow \overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ is surjective. Furthermore, since the curve $[C^{\text{cusp}}(2)]$ is not \mathcal{Z} -stable, we have the inclusion

$$V^{\mathcal{Z}} \subset \tilde{\mathcal{U}}_{2,2}^{ns}((1, 1), (3, 0)) \setminus Z.$$

Thus, the restriction of the map (1.2.3) gives us a regular morphism

$$V^{\mathcal{Z}} \rightarrow \overline{\mathcal{U}}_{2,1}^{ns}(2), \tag{2.4.1}$$

contracting $\widetilde{\mathcal{W}}$ to a point.

Now, similarly to the proof of Theorem A we check that the morphism (2.4.1) factors through $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$. Note that to apply the same argument as in Theorem A we use the following facts: (i) $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ is smooth (see Corollary 2.4.2); (ii) the projection $V^{\mathcal{Z}} \rightarrow \overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ is smooth (since p_2 varies in a smooth part of a curve); and (iii) $V^{\mathcal{Z}} \times_{\overline{\mathcal{M}}_{2,1}(\mathcal{Z})} V^{\mathcal{Z}}$ is irreducible, as a \mathbb{G}_m^3 -torsor over the moduli stack of (C, p_1, p_2, p'_2) with C smoothable.

This gives us the required morphism ϕ_2 contracting to $\overline{\mathcal{W}}$ to some point in $\overline{\mathcal{U}}_{2,1}^{ns}(2)$. On the other hand, by Proposition 2.4.1, the only point in $\overline{\mathcal{U}}_{2,1}^{ns}(2)$, which is not \mathcal{Z} -stable is $[C_0]$ (recall that by this we mean the pointed curve (C_0, p) , where $p \neq 0, \infty$, see Lemma 2.2.1). Thus, the rational map ϕ_2^{-1} is regular on $\overline{\mathcal{U}}_{2,1}^{ns}(2) \setminus [C_0]$ (and sends (C, p) to (C, p)). Also, the restriction of ϕ_2 to $\overline{\mathcal{M}}_{2,1}(\mathcal{Z}) \setminus \overline{\mathcal{W}}$, i.e., to the locus where $h^0(2p) = 1$, is an open embedding sending (C, p) to (C, p) . This implies that $\phi_2(\overline{\mathcal{W}}) = [C_0]$, and ϕ_2 induces an isomorphism of $\overline{\mathcal{M}}_{2,1}(\mathcal{Z}) \setminus \overline{\mathcal{W}}$ with $\overline{\mathcal{U}}_{2,1}^{ns}(2) \setminus [C_0]$. \square

Let us consider the natural birational maps of the coarse moduli spaces

$$\overline{\mathcal{M}}_{2,1} \dashrightarrow \overline{\mathcal{M}}_{2,1}(\mathcal{Z}) \dashrightarrow \overline{\mathcal{U}}_{2,1}^{ns}(2).$$

Note that all these spaces are normal (for the last two this follows from Proposition 2.1.1 and Corollary 2.4.2). Note also that we only know that $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ is a proper algebraic space.

Let $\overline{\mathcal{W}} \subset \overline{\mathcal{M}}_{2,1}$ denote the closure of \mathcal{W} , and let $\Delta_1 \subset \overline{\mathcal{M}}_{2,1}$ be the boundary divisor, whose generic point corresponds to the union of two elliptic curves.

Proposition 2.4.6. *The natural birational morphism $f : \overline{\mathcal{M}}_{2,1} \dashrightarrow \overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ (resp., $g : \overline{\mathcal{M}}_{2,1} \dashrightarrow \overline{\mathcal{U}}_{2,1}^{ns}(2)$) is a birational contraction with the exceptional divisor Δ_1 (resp., exceptional divisors Δ_1 and $\overline{\mathcal{W}}$).*

Proof. Recall that to check that f (resp., g) is a birational contraction we need to check that the exceptional locus $\text{Exc}(f^{-1})$ (resp., $\text{Exc}(g^{-1})$) has codimension ≥ 2 . But this immediately follows from Lemma 2.4.3. Next, the restriction of f to the complement of Δ_1 induces an isomorphism with the open subset in $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ consisting of (C, p) with C smooth or nodal, so we have an inclusion $\text{Exc}(f) \subset \Delta_1$. On the other hand, the generic point of Δ_1 corresponds to the union of elliptic curves $E_1 \cup E_2$, with the marked point on E_1 . Under the map f this curve gets replaced by the cuspidal curve \overline{E}_1 , so that we have a contraction $E_1 \cup E_2 \rightarrow \overline{E}_1$ sending the elliptic tail E_2 to the cusp. Since this map forgets the j -invariant of E_2 , this means that Δ_1 gets contracted by f . Now the fact that $\text{Exc}(g) = \Delta_1 \cup \overline{W}$ follows from Theorem 2.4.5. \square

Remark 2.4.7. Let $\overline{\mathcal{V}}_{g,1}^{ns}(g) \subset \overline{\mathcal{U}}_{g,1}^{ns}(g)$ be the irreducible component consisting of smooth-able curves. Theorem A implies that the natural birational map

$$\overline{\mathcal{M}}_{g,1} \dashrightarrow \overline{\mathcal{V}}_{g,1}^{ns}(g)$$

contracts \overline{W} to a point. Passing to the normalizations of the coarse moduli spaces we get the birational map $\phi : \overline{\mathcal{M}}_{g,1} \dashrightarrow X$, where X is a normal projective variety, contracting \overline{W} to a point. It seems plausible that ϕ is a birational contraction (which would imply that \overline{W} is an extremal divisor). To check this we would need to prove that $\text{Exc}(\phi^{-1})$ has codimension ≥ 2 . In other words, we would need to check that the locus in $\overline{\mathcal{V}}_{g,1}^{ns}(g)$, consisting of unstable (i.e., non-nodal) curves, has codimension ≥ 2 . In the case $g = 2$ we have shown this in Lemma 2.4.3. Note that the fact that the class of \overline{W} generates an extremal ray in $\overline{NE}^1(\overline{\mathcal{M}}_{g,1})$ is known for $g \leq 3$ and $g = 5$, by the works [9], [5] and [6].

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